Surface charges and fields of simple circuits

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Interest in the surface charges on circuits, and their utility in the conceptual understanding of circuit behavior, has recently increased. Papers and textbooks have discussed surface charges either with qualitative diagrams or analytic results for very special geometries. Here, I present the results of numerical calculations showing the surface charges on several simple resistor-capacitor circuits. Surface charges are seen to guide the motion of charges and create the appropriate electric potential and Poynting vectors for the circuit, and hence are an important factor in the teaching of circuit theory.

I. INTRODUCTION

Interest in surface charges on circuits has been revitalized recently by the introductory textbook by Chabay and Sherwood[1] (hereafter CS). They use the concept of surface charges to help students understand the electric fields and currents in a circuit, including transients that occur when one opens or closes a switch (pedagogical details are given in Sherwood and Chabey[2,3]). Jackson[4] presented three roles for the surface charges, discussed in Sec. V. Actual calculations of the surface charge have not been available except for very special geometries (see Sec. II).

The present paper presents numerical calculations of surface charges for several circuits, corresponding to qualitative examples in CS (Chap. 6). The charge distribution is found by a relaxation technique, which determines the self-consistent distribution of charges (and the resulting fields) for the circuits. These charges and fields result in the same current moving throughout the circuit. Details of the calculations are in Sec. III and results are presented in Sec. IV.

The modeled circuits are resistor-capacitor (*RC*) circuits, rather than resistor-battery circuits, to eliminate the difficulties modeling a battery with its nonconservative forces.[5,6] While a *RC* circuit has an equilibrium current of zero, it nevertheless can have an exponential decay which lasts much longer than the time for computational transients to die out. Circuits with capacitors as the charge source are also used in CS (Chap. 7) as both pedagogical and laboratory exercises, and these calculations can also apply to those cases.

II. PRIOR WORK

Only a few circuit geometries are amenable to analytic solutions: the infinite straight wire or coaxial wires,[7] finite coaxial wires,[4] circular loops,[8] a spherical battery,[6] and a 'squared coil'[9] or ring[10] in magnetic fields. None of these circuits looks very much like the ones in introductory textbooks.

Rosser[11] gives a short calculation showing the small number of charges needed to guide current through a bend. Sherwood and Chabay[3] give an extensive review of this literature.

Analytic solutions are found by solving Laplace's equation, which gives time-independent results. These solutions cannot, therefore, show the transient behavior of the circuit before the establishment of steady state (which is important pedagogically[2,3,12–14]).

Some lecture demonstrations have been published which show the electric fields[15] and surface charges (detected with an electroscope)[16,17] around a circuit. These demonstrations allow arbitrary one- and two-dimensional circuits, with resistors, capacitors, and batteries, but cannot show transient responses, which are a few light-crossing times.

Some computer calculations have been done to illustrate the peculiarities of a $1/r^2$ force law,[5] and to augment analytic calculations.[4] White, Frederiksen, and Spoehr[18] used a computer simulation of a ''transport model'' of charges in a circuit to study the effectiveness of various conceptual models in teaching electric circuits.

Jefimenko's textbook[19] was perhaps the first to recognize and discuss surface charges in an introductory text (see also his answer to a student question about current flow[20]). Härtel[21,22] discussed the pedagogical importance of surface charges in circuits. Swartz,[23] Swartz and Miner,[24] and Griffiths[25] are among the texts that discuss the role of surface charges and feedback in circuits. Chabay and Sherwood[1] have produced an excellent introductory textbook using surface charges to link electrostatics and circuit concepts, rather than the typical text which treats these as disjoint topics.

III. DETAILS OF THE CALCULATIONS

Four different *RC* circuits were modeled. All had parallel plate capacitors 16.5×5.25 mm on a side, separated by 1.0 mm. The plates were hooked together with copper wires, 5.25 mm in cross section. The circuits differed in the configuration of the wires, following examples in CS[1] (pp. 208–219).

Uniform resistive wire: A single fat copper wire connects the capacitor plates. This is the simplest circuit, and shows various polarization effects as well as the ''guiding charges'' for the current (see Fig. 1 and Sec. IV A).

Lumped resistor: This circuit has a region with one-tenth the conductivity of copper (a resistor). This model illustrates the surface charges which occur when there is a discontinuous change in the conductivity (see Fig. 5 and Sec. IV B).

Narrow wire: This circuit has a region with a narrow copper wire. This model illustrates surface charges piling up on either side of a resistor to create uniform current flow (see Fig. 7 and Sec. IV C).

Sinuous wire: This circuit has a single copper wire snaked in a sinuous path. This model illustrates the global nature of the circuit and the strongly coupled nature of the charges.[26] In one portion of the circuit, the current must flow in the direction opposite to that of the dipole electric field of the

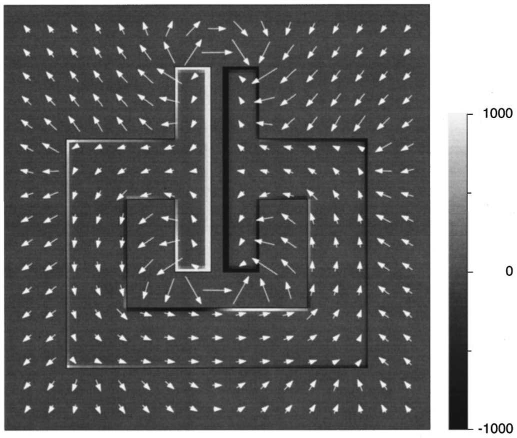

Fig. 1. The diagram is a cross section through the midplane of the circuit. The white to black shades represent excess charges, ranging between $\pm 1000\, e/\mathrm{mm}^3$ (the charges on the capacitor exceed this range, which was chosen to illustrate the surface charges on the wires). The arrows plot the square root of the electric field. The large field vectors between the plates have been omitted.

capacitor. The development of steady-state current flow is pedagogically interesting (details to be published elsewhere[27]). See Fig. 8 and Sec. IV D.

The circuit is assumed to obey a Drude model: The wires are filled with equal densities of positive and negative charges of magnitude e, and the local current density is related to the electric field and the conductivity σ,

$$\mathbf{J}=\sigma\mathbf{E}. \tag{1}$$

The circuit is divided into cubic computational cells 0.25 mm on a side, and contains approximately $100\times100\times20 = 2\times10^5$ cells. Any excess charge is assumed to reside in the center of each cell. The electric field is calculated at the center of each face of a cell by Coulomb's law,

$$\mathbf{E}=\sum_i \frac{q_i\mathbf{r}_i}{4\pi\epsilon_0 r_i^2}, \tag{2}$$

where r_i is the distance between the center of the face and the center of cell i. This is done by direct summation, rather than one of the tree codes[28] or other high-powered techniques, because the number of cells is relatively small. This simple technique also means that freshman physics students can understand how the calculations were performed.

Charges are then moved from one cell to another across the face, thus conserving charge, according to Eq. (1) (multiplied by $s^2\Delta t$),

$$\Delta q=\sigma E_n s^2 \Delta t, \tag{3}$$

where E_n is the normal component of $\mathbf{E}$ at this surface, s^2 is the area of the face ($[0.25\ \mathrm{mm}]^2$), and $\Delta t = 5\times10^{-20}$ s. This very small time was chosen so the fractional change in the charge of a cell was small, typically less than 1%.

The calculation is started by placing $\pm 10^5 e/\mathrm{mm}^2$ on the inner faces of the capacitor, with the rest of the circuit neutral. The program makes a loop over all the cells in the circuit, calculating the electric field on each cell face and moving charges based on that field and the conductivity. The process is then repeated with the new charge configuration. The figures are shown after approximately 200 such steps (each step requiring about 1 h on three dual-processor Pentium computers, parallelizing the calculation). The calculations are stopped when the electric fields reach steady state (i.e., when the electric fields and charges have stopped changing, except for the slow decay on the time scale RC). Extending the calculation results only in the capacitor slowly discharging and all the electric fields and surface charges slowing decreasing to zero.

This calculation can either be viewed as a relaxation technique (where the charges are moved to produce a self-consistent electric field) or as a time-dependent calculation. As mentioned before, the equilibrium current in a RC circuit is zero, but the circuit is in exponential decay far longer than the time for the calculation to relax. The short relaxation time makes the concept of a relaxed solution with nonzero current meaningful.

Viewing the calculations as the time response of the circuit presents problems because retardation effects have been ignored: The charges in Eq. (2) are not the charges at a time r/c earlier than now, but are the charges *right now*, and so all transient effects are suspect (see Härtel,[22] pp. 16 and 17, for a discussion). Once steady state has been reached, however, the differences between one step and the next can reasonably be taken as reliable, and these differences are used for the calculation of the displacement current [Eq. (5)].

The magnetic field and Poynting vector were also calculated for these systems. The magnetic field was calculated using the Biot–Savart law,

$$\mathbf{B}=\frac{\mu_0}{4\pi}\oint\frac{\mathbf{J}\times\mathbf{r}}{r^2}\,d\tau \tag{4}$$

with $\mathbf{J}=\mathbf{J}_c+\mathbf{J}_d$, the sum of real and displacement current densities.[29] The real current density is found from Eq. (2) and the displacement current density from

$$\mathbf{J}_d=\epsilon_0\frac{\partial\mathbf{E}}{\partial t}. \tag{5}$$

The time derivative is found by the difference between two steps of the fully relaxed solution, when the computational transients have passed and we have just the slow decay of the capacitor's charge.

The range of magnitudes of the electric field and Poynting vectors is too great to easily plot, so the square root of the magnitude of the vectors is plotted in the figures. The very large and uninteresting electric fields between the capacitor plates are not plotted.

Ideally, the charges should reside entirely on the surface of the conductors, assuming steady state and no discontinuities in the conductivity. The calculated circuits, however, have small amounts of charge in the interior of the wires. These charges result from the unrealistic discreteness in the position of the charges, caused by the necessity of employing a finite number of cells. Other calculations (not shown) show that the charge density falls by about one order of magnitude for each cell inward from the surface, and so higher resolution studies would have even less unphysical charge in the interior of the wires.

IV. RESULTS

A. Uniform resistive wire

This is the simplest RC circuit examined (see Fig. 1). Within limitations due to finite-size effects, all the charges are on the surfaces of the conductors. The electric field of the

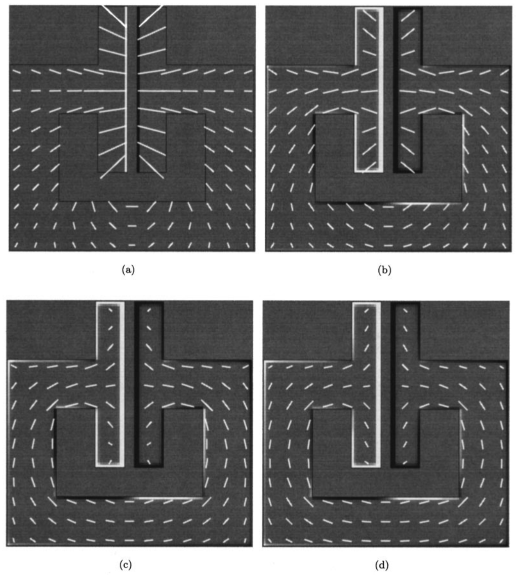

Fig. 2. These diagrams illustrate the relaxation of the solution, from the initial conditions in (a) to the steady-state solution in (d). The panels are after 0, 10, 40, and 160 steps. The scale is the same as Fig. 1. The lines plot the square root of the electric field inside the wires, and the arrowheads have been omitted for clarity.

capacitor is basically dipolar, but the surface charges modify the field inside the wires so the field is uniform in magnitude and parallel to the wires, thus creating a uniform flow of charges.

Figure 2 shows the progress of the relaxation solution from the initial configuration [Fig. 2(a)] to steady state [Fig. 2(d)]. The figures are plotted after 0, 10, 40, and 160 steps. These figures should not be taken as the actual transient response of the circuit (see Sec. III), but do help isolate specific features.

Initially, the electric field is just that of a finite dipole. This electric field causes large polarization charges on the inner and outer surfaces of the wires [Fig. 2(b)]. Note, too, that the charges do not remain on the inner surfaces of the capacitors, but some move to the outer surfaces due to the fringe field of the finite capacitor. This fringe field drives charges away from the plates and around the circuit (Ref. 1, pp. 140–143).

The remaining steps to steady state involve primarily accumulating surface charges in the corners of the circuit (these charges then deflect current from the corners) and producing the appropriate gradient of charge to modify the electric field in the wires. The net effect is that the electric field in the wires changes from the dipolar field of the capacitor to a field with uniform magnitude everywhere parallel to the wires (details of the feedback process are in Preyer[27]).

Figure 3 plots the equipotentials for this system. Note the even spacing of equipotentials (appropriate to uniform conductivity), and compare with the Poynting vectors in Fig. 4.[8] Since the electric field is the plane of the figure, and the magnetic field is perpendicular to the plane, **S** must lie in the plane and be perpendicular to **E**, and hence parallel to the equipotential lines. Note the energy flow is from the capacitor to the resistive wires, as expected.

B. Lumped resistor

This circuit (Fig. 5) has a lumped resistance: a region with a conductivity one-tenth that of the copper wires, located in the middle of the bottom wire. Unlike the other models, this example has excess surface charges at the interior boundary where the conductivity changes (this result comes from applying Gauss's law across the boundary). These surface charges resemble those of a parallel-plate capacitor, and increase the electric field in the low-conductivity region and decrease the field in the high-conductivity region. This results in equal current flow through all parts of the circuit. Plots of the equipotentials (Fig. 6) show the much greater potential drop across the resistive region. As discussed above, the equipotential lines are parallel to the Poynting vectors, so we also see the much larger energy flow into the resistive region.

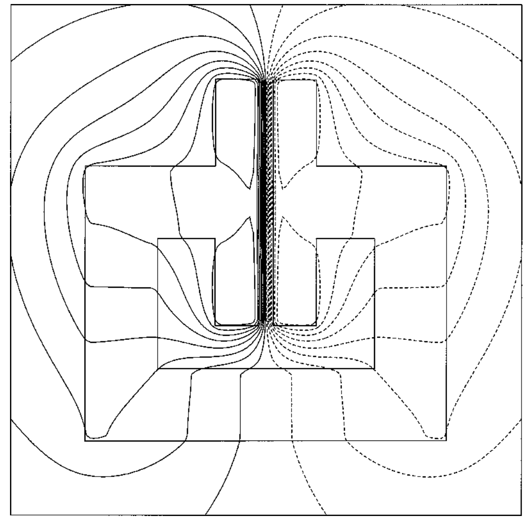

Fig. 3. A plot of the equipotentials (from +2.5 to −2.5 V in steps of 1/3 V). Positive (negative) equipotentials have a solid (dashed) line.

C. Narrow wire

This circuit (Fig. 7) has high-conductivity copper wires throughout, but a narrowed region in the bottom wire (the

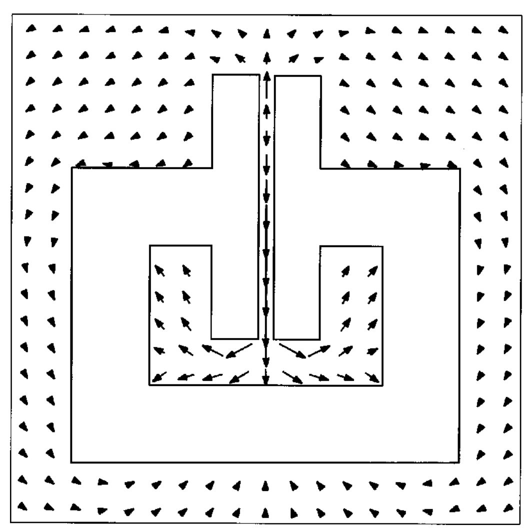

Fig. 4. A plot of the Poynting vectors in the exterior of the circuit (the Poynting vector is nonzero inside the wires, but this is not shown). The arrows plot the square root of the magnitude of the vectors. The text discusses the relationship of the Poynting vectors with the equipotential lines of Fig. 3.

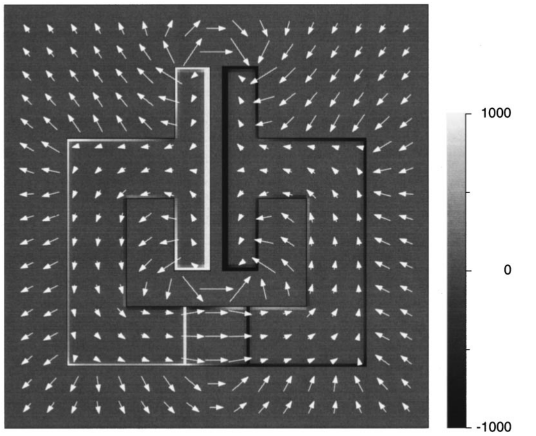

Fig. 5. The diagram is a cross section through the midplane of the circuit of Fig. 1, but with a resistive region in the bottom wire. The white to black shades represent excess charges, ranging between $\pm 1000\, e/\mathrm{mm}^3$ (the charges on the capacitor exceed this range, which was chosen to illustrate the surface charges on the wires). The arrows plot the square root of the electric field. The large field vectors between the plates have been omitted.

wire is narrow out of the plane of the page, as well). Again, capacitor-like charges form, increasing the electric field (and hence current density) through the narrow region and decreasing the field and current density in the wider wires.

D. Sinuous wire

All the previous circuits basically drive current in the direction of the dipolar field of the capacitor. This circuit (Fig. 8) is of interest because of the highlighted region, where current is flowing in a direction opposite to that of the dipolar field. CS (Ref. 1, pp. 208–210) use this example to illustrate the importance of the surface charges: Besides fine-tuning the electric field of the source, they can also reverse the field direction completely. Pedagogical details of the equilibration of this circuit will be presented elsewhere.[27]

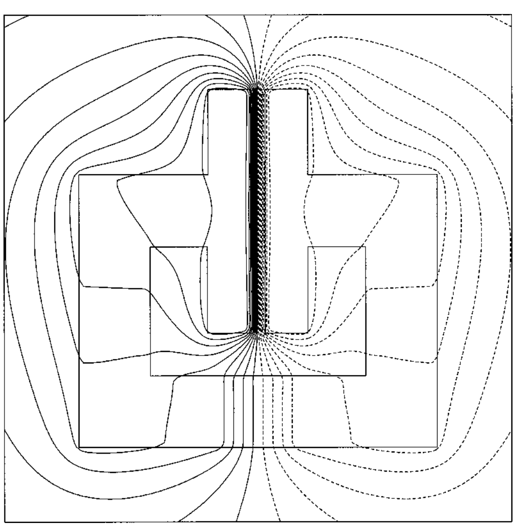

Fig. 6. A plot of the equipotentials (from +2.5 to −2.5 V in steps of 1/3 V) for the lumped resistor circuit. Positive (negative) equipotentials have a solid (dashed) line.

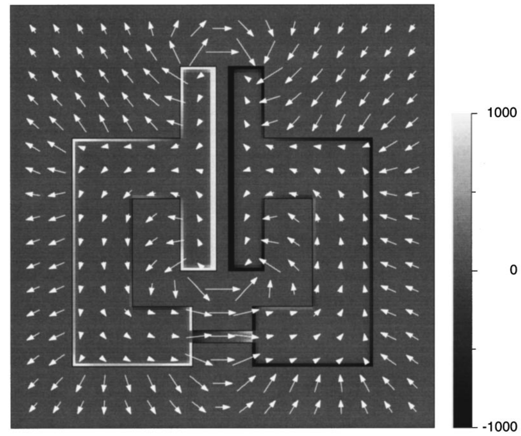

Fig. 7. The diagram is a cross section through the mid-plane of the narrow wire circuit. All the wires have the same conductivity. The white to black shades represent excess charges, ranging between $\pm 1000\, e/\mathrm{mm}^3$ (the charges on the capacitor exceed this range, which was chosen to illustrate the surface charges on the wires). The arrows plot the square root of the electric field. The large field vectors between the plates have been omitted.

V. DISCUSSION

Jackson[4] describes the three roles of surface charges in circuits: ''(1) to maintain the potential around the circuit, (2) to provide the electric field in the space around the circuit, (3) and to assure the confined flow of current.'' Within the limitations of finite-size effects, the present calculations illustrate these three roles.

(1) As Figs. 3 and 6 show, the equipotential surfaces behave reasonably: a small gradient where the conductivity is

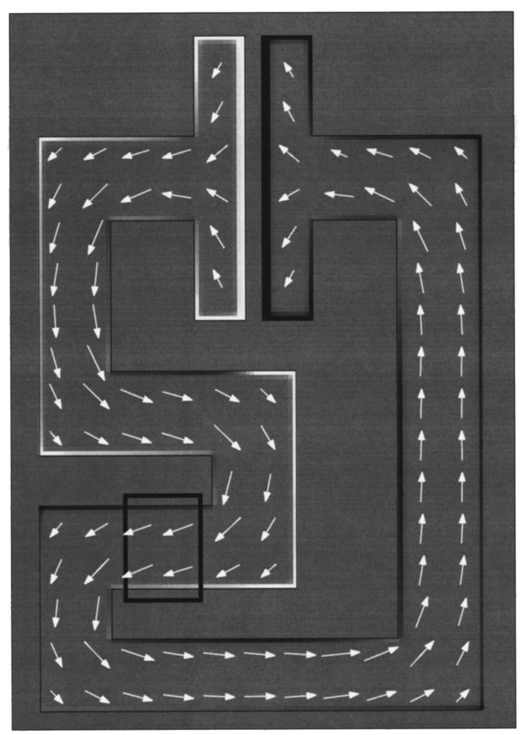

Fig. 8. The diagram is a cross section through the mid-plane of the sinuous wire circuit. All the wires have the same conductivity. The arrows plot the square root of the electric field. All field vectors external to the circuit have been omitted.

large, and a large gradient where the conductivity is small. The surfaces are everywhere perpendicular to the motion of current.

(2) The electric field outside the conductors is important for ensuring that the Poynting vectors point in the correct directions, and that high-resistance regions have a greater energy flow than low-resistance regions (see Figs. 4 and 6).

(3) The electric field inside the conductors is modified (see Fig. 2) to be everywhere parallel to the wires. In wires of constant conductivity and width, the internal electric field is uniform in magnitude. In other situations the surface charges increase the electric field in high-resistance regions, and decrease the field in low-resistance regions until, by a feedback process, the current has the same value in all segments.

These roles are the key for a qualitative understanding of circuits, and the calculation of quantitative pictures of the surface charges can only increase student comprehension.

A major limitation of this work is the lack of retardation effects, which makes the calculation of the transient response impossible. This limitation will be addressed in future work.

Large color versions of these and other figures are available at my web site, http://galaxy.cofc.edu/circuits.html. The computer codes are also available upon request.

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